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LETTER TO THE EDITOR

Superfluid quark matter

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Abstract. Assuming that ultra-dense matter behaves as a fluid of quarks rather than hadrons, we investigate the possible superfluid order parameters which may arise.

It has been suggested (Collins and Perry 1975) that at densities of the order of one baryon per cubic fermi, matter will exist as a system of quarks rather than a system of hadrons. The reason for this is that the long-range forces which normally confine quarks to hadrons are screened by the many-body effects of the medium. Attempts to calculate the density at which the hadronic matter to quark matter phase transition occurs have been made by a number of authors with results in the range 10^{14}–10^{15} g cm^{-3}. (See, for example, Baluni 1978, and references therein.) The significance of this observation is that densities of this order may be attained in neutron star cores, and in the early universe. They may also be attainable in the laboratory in collisions between heavy ions, since ordinary nuclear matter has a density of about 10^{14} g cm^{-3}. In this Letter we investigate possible superfluid order parameters for quark matter; Cooper pairs of quarks may be formed by the short-range force between them, in the presence of the fermi sea of other quarks.

The current theory of strong interactions (see Politzer 1974 for a review) contains three colours of quarks, and an undetermined number of flavours (u, d, s, c, b, etc). The interaction between quarks is by the exchange of coloured vector gluons which take no account of flavour. The effective strong interaction coupling constant g_s decreases with distance, and at the separations between quarks with which we are concerned here (less than one fermi), experience with deep inelastic electron and neutrino scattering indicates that g_s is small enough for the use of lowest-order perturbation theory. Accordingly, we take the effective interaction between two quarks in free space from the one-gluon exchange diagram (figure 1). (We assume that non-perturbative instanton effects are unimportant at these distances (Applequist and Shankar 1978).) If we are able to describe the situation by a potential between the quarks, then in the non-relativistic limit its form will be

$$V_{ki,lj} = \frac{\alpha_s}{r}(t_a)_{ki}(t_a)_{lj}, \tag{1}$$

where $\alpha_s = g_s^2/4\pi$ is the strong fine structure constant, t_a $(a = 1, \ldots, 8)$ are the generators of the colour group for quarks, and k, i, l, j are the colour indices of the quarks. At order m^{-2}, where m is the quark mass, there are also spin-dependent

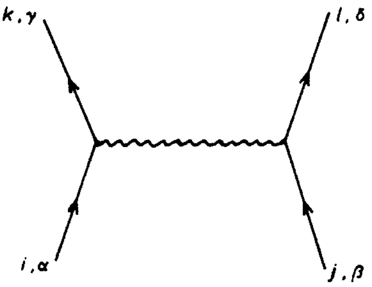

Figure 1. One-gluon exchange diagram.

terms, but the structure in colour space is as above. Since

$$(t_a)_{ki}(t_a)_{lj} = \tfrac{1}{6}(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) - \tfrac{1}{3}(\delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}), \tag{2}$$

there is a repulsive force in the channel symmetric in the colour indices (corresponding to the six-dimensional representation of colour SU(3)) and an attractive force in the channel antisymmetric in the colour indices (corresponding to the $\bar{3}$ representation of colour SU(3)).

If we estimate the many-body effects of the medium of quarks using random-phase approximation (ring diagram approximation), then the structure of the interaction in colour space is preserved but the interaction is screened (see Fetter and Walecka 1971). Consequently, we will expect Cooper pairs belonging to the $\bar{3}$ of colour SU(3) to be formed (since any attractive interaction, no matter how weak, forms bound states in the presence of the fermi sea).

In what follows, we proceed in a non-relativistic fashion. Our justification for this is that the distances between quarks in quark matter may be comparable to those in a hadron, if the estimates of the density at which the transition to quark matter occurs are reliable (Baluni 1978 and references therein). Also, the non-relativistic quark model approach has proved successful in discussions of the properties of hadronic states, despite the fact that the typical quark momentum is comparable with the quark mass. (See, e.g. De Rujula *et al* 1975.) The weakness in our argument is that we are not sure what mass to assign to quarks in quark matter. Non-relativistic quark models of hadronic states give the u and d quarks masses of about 350 MeV, but discussions of chiral symmetry breaking, which concentrate on quark mass terms in the Lagrangian, give masses of not more than 100 MeV, and perhaps only tens of MeV. Which mass is the correct one to use here depends on the difficult and unresolved question of whether the transition from hadronic to quark matter in which the quarks become unconfined produces a discontinuity in the effective quark masses. If it does, we may need quark masses of as little as tens of MeV. If not, then since we have in mind densities close to the density of a hadron, the u and d quark masses may differ little from 350 MeV. We proceed on this latter assumption. In that case it makes sense to classify the quark Cooper pairs by an orbital angular momentum, L, and a spin angular momentum, S.

The effective quark mass, m^*, in the presence of the medium of quarks, may differ from m because of many-body effects. However, if we use the Coloumb-like potential of equation (1), and calculate in random-phase approximation, then a calculation analogous to that for the degenerate electron gas (Quinn and Ferrell 1958) gives

$$m/m^* = 1 - r_s(4/3\pi)^{1/3}(2/3\pi)[2 + \ln(4/3\pi)^{1/3}(r_s/2\pi)] \tag{3}$$

with the approximation being valid for $r_s < 1$, where r_s is the interquark spacing divided by the strong Bohr radius, $\hbar/mc\alpha_s$.

For $0 < r_s < 1$, m/m^* differs very little from 1. (For $m \simeq 350$ MeV, and $\alpha_s \simeq 0{\cdot}3$, and an interquark spacing of 0·3 fermi, we have $r_s \simeq 0{\cdot}2$.)

In the following, weak-coupling BCS theory is used. This is justified because the range of energy ΔE over which the screened Coulomb-like potential between quarks varies is on the scale of $r_s E_F$, so that

$$\Delta E/k_B T_c \simeq r_s(T_F/T_c). \tag{4}$$

Provided r_s is not too small, and $T_F/T_c \simeq 10^3$, as for superconductors and ^{3}He, $\Delta E/k_B T_c \gg 1$, and weak-coupling theory is a good approximation. We observe in passing that the size of the quark Cooper pair ($\xi_0 = \hbar v_F/\pi k_B T_c$) will be several hundred fermi, compared with a range of the interquark force of the order of a fermi. Thus, just as for ^{3}He (see e.g. Leggett 1975), the particles in the Cooper pair spend most of their time outside the range of the pairing force.

In the quark matter core of a neutron star there will be approximately twice as many d quarks as u cs. Thus the fermi energies for the two types of quark will be very different and we do not expect Cooper pairing of a u quark with a d quark. We therefore restrict attention, in the first instance, to pairing between quarks of the same flavour. In that case, the flavour wavefunction is symmetric, and since the colour $\bar{3}$ wavefunction is antisymmetric, fermi statistics allow $S = 1$, L even, and $S = 0$, L odd. Assuming that, as for hadronic states, the lowest value of L is the most tightly bound, we shall consider $S = 1, L = 0$ and $S = 0, L = 1$. If experience with hadronic states is a good guide (see e.g. De Rujula *et al* 1975), $S = 1$, $L = 0$ may be the case which is preferred physically. However, we err on the side of caution and consider both cases.

Consider first $S = 1$, $L = 0$ Cooper pairs. The superfluid order parameter is

$$\psi_{ij,\alpha\beta} = \sum_{|\boldsymbol{k}|} \langle a_{i\alpha}(-\boldsymbol{k}) a_{j\beta}(\boldsymbol{k})\rangle \tag{5}$$

where, in the many-body theory, $a_{i\alpha}(\boldsymbol{k})$ annihilates a quark with colour index i, spin index α and wavevector $\boldsymbol{k}$. For colour $\bar{3}$, $S = 1$, Cooper pairs we write

$$\psi_{ij,\alpha\beta} = \epsilon_{ijk}(i\sigma_a\sigma_2)_{\alpha\beta}\bar{d}_{ka} \tag{6}$$

to obtain an order parameter $\bar{d}_{ka}$ which transforms as colour $\bar{3}$ on the row index k and as spin triplet on the column index a. Since we want to consider $L = 0$, $\bar{d}_{ka}$ has no dependence on $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}|$.

With the aid of a Bogoliubov transformation (Bogoliubov 1958, Valatin 1958) we may derive the superfluid free energy density in the weak-coupling BCS approximation. In the Ginzburg–Landau region it takes the form

$$F = \tfrac{1}{2}\frac{dn}{d\epsilon}\frac{(T - T_c)}{T_c} tr(d^\dagger d) + \tfrac{7}{32}\frac{dn}{d\epsilon}(\pi k_B T_c)^{-2}\zeta(3)[(tr\, d^\dagger d)^2 + tr(d^\dagger d)^2 - tr(d^\dagger d)(d^\dagger d)^*]$$
$$- \tfrac{93}{256}\frac{dn}{d\epsilon}(\pi k_B T_c)^{-4}\zeta(5)[\tfrac{2}{3}(tr\, d^\dagger d)^3 + \tfrac{4}{3}tr(d^\dagger d)^3 - tr(d^\dagger d)tr(d^\dagger d)(d^\dagger d)^*], \tag{7a}$$

where

$$d = [\tfrac{1}{2}\frac{dn}{d\epsilon}\ln(1{\cdot}14\epsilon_0/k_B T_c)]^{-1}\bar{d} \tag{7b}$$

with ϵ_0 the effective cut-off of the pairing interaction, and we have retained up to sixth order in the order parameter. The quartic terms in the free energy are minimised by any order parameter with $d^\dagger d$ real. This degeneracy of superfluid phases is resolved by the sixth-order terms which uniquely select the phase represented by

$$d = I/\sqrt{3}, \tag{8}$$

where I is the identity 3×3 matrix. (For convenience in determining possible phases, we have changed the normalisation of d so that $tr\, d^\dagger d = 1$.)

By analogy with superfluid ^{3}He we shall refer to this as a B-like phase, though the rows and columns of the matrix refer to colour and spin, rather than spin and orbital angular momentum, and the symmetries of the free energy differ correspondingly. In the present case, the free energy is invariant if the order parameter is multiplied on the left by a special unitary matrix, or on the right by a special orthogonal matrix, or multiplied by a phase factor. Any order parameter

$$d = U/\sqrt{3}, \tag{9}$$

where U is a general unitary matrix, represents the same phase as equation (8).

More generally, the symmetries of the free energy require the quartic terms to take the form

$$F_4 = \beta_2(tr\, d^\dagger d)^2 + \beta_4 tr(d^\dagger d)^2 + \beta_3 tr(d^\dagger d)(d^\dagger d)^*. \tag{10}$$

Depending on the relative values of β_3 and β_4, various phases are possible as shown in figure 2. (The techniques of Barton and Moore (1974) are useful here.) By the polar-like phase we mean the phase with $d_{11} = 1$ and all other entries zero, and by the A-like phase we mean the phase with $d_{11} = 1/\sqrt{2}$, and $d_{12} = i/\sqrt{2}$, and all other entries zero. (In the present context, the A-like phase and the A_1-like phase are equivalent because of the unitary symmetry of the free energy.) It is possible that corrections to the weak-coupling BCS values $\beta_2 = \beta_4 = -\beta_3$ cause the polar-like phase to be lower in free energy than the B-like phase. We have not carried out the lengthy, and not necessarily dependable, calculation of Landau parameters necessary to answer this question. If this does happen, there may be more than one superfluid phase transition, since, as

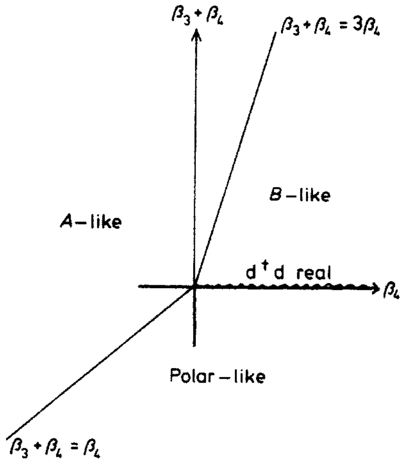

Figure 2. Phase diagram for $S = 1$, $L = 0$ superfluid quark matter.

discussed above, the sixth-order terms in the free energy are minimised by the B-like phase, and these will become important at sufficiently low temperatures.

In the case of $S=0$, $L=1$ Cooper pairs, we have to consider the superfluidfr parameter

$$\psi_{ij}=\sum_{|k|}\langle a_{i\uparrow}(-\boldsymbol{k})a_{j\downarrow}(\boldsymbol{k})\rangle, \tag{11}$$

where i and j are colour indices, as before. For colour $\bar{3}$ we write

$$\psi_{ij}=\epsilon_{ijk}d_k, \tag{12}$$

and since we are considering p-wave Cooper pairs we write

$$d_i=A_{i\rho}n_\rho, \tag{13}$$

where

$$\boldsymbol{n}=\boldsymbol{k}/|\boldsymbol{k}| \tag{14}$$

and $A_{i\rho}$ has no $\boldsymbol{n}$ dependence.

$A_{i\rho}$ transforms as colour $\bar{3}$ on its row index and as an orbit space vector on its column index.

In the Ginzburg–Landau region, the most general set of quartic terms allowed by the symmetry of the free energy (under colour and orbital space transformations and multiplication of $A_{i\rho}$ by a phase factor) is

$$F_4=b_2(trA^\dagger A)^2+b_4tr(A^\dagger A)^2+b_3tr(A^\dagger A)(A^\dagger A)^*. \tag{15}$$

The situation is exactly analogous to the one discussed above for $S=1$, $L=0$ (cf equation (10)) and the phase diagram has the same form with the matrix $A_{i\rho}$ replacing the matrix $d_{i\alpha}$. However, the weak-coupling BCS values of the parameters are in this case

$$b_2=b_3=b_4=\frac{7}{15}\frac{1}{64}\frac{dn}{d\epsilon}(\pi k_B T_c)^{-2}\zeta(3). \tag{16}$$

Thus the weak-coupling values uniquely choose a B-like phase (with the rows of the matrix signifying colour, and the columns signifying orbital coordinates) without appealing to the sixth-order terms. In this case it seems unlikely that the order T_c/T_F strong-coupling corrections will change the conclusion.

Although Cooper pairing of u quarks to d quarks is not likely to occur in neutron star cores, as discussed above, it may be possible to produce such pairing in heavy-ion collisions, by using ions with an equal total number of protons and neutrons and hence of u quarks and d quarks. Given that the Cooper pairs are colour $\bar{3}$, the possibilities of lowest orbital angular momentum consistent with fermi statistics are

$$(I=0, S=0, L=0), \qquad (I=0, S=1, L=1),$$
$$(I=1, S=1, L=0) \quad \text{and} \quad (I=1, S=0, L=1),$$

where I denotes isospin. Experience with hadronic states suggests that the most tightly bound will be $(I=0, S=0, L=0)$. In that case, the order parameter is a three-component object transforming as a $\bar{3}$ of colour, which we may choose to have its first entry non-zero and all other entries zero.

Quark matter superfluidity in a neutron star core can affect the angular momentum of the neutron star through the motion of vortex lines, just as for neutron matter superfluidity. It can also affect the cooling of the star through its effect on neutrino emission (Maxwell *et al* 1977) and the order parameters we have discussed may be useful as an input to detailed astrophysical calculations. On the other hand, at the densities necessary for quark matter to exist, the early universe is likely to be too hot for superfluidity (if we estimate $T_c \simeq 10^{-3}\ T_F$.)

Perhaps the best chance of a direct test of quark matter superfluidity is in heavy-ion collisions (although there will be complications due to finite size effects; in particular, from the size of the Cooper pairs). It may perhaps be possible to study the superfluid order in spin space by using hadrons to probe the quark matter system. It may also be possible to use techniques analogous to the spin-resonance techniques which have been so useful in the case of superfluid ^{3}He (see e.g. Leggett 1975). At order m^{-2}, where m is the quark mass, there are terms arising from one-gluon exchange (figure 1) which couple spin space to orbit space, and could in principle lead to spin-resonance effects, in the case when L and S are both non-zero. For example, there is a tensor potential

$$V_T = \frac{3}{4m^2}\frac{\alpha_s}{r^3}\left[\frac{(\boldsymbol{\sigma}^1 \cdot \boldsymbol{r})(\boldsymbol{\sigma}^2 \cdot \boldsymbol{r})}{r^2} - \tfrac{1}{3}\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2\right] \tag{17}$$

(see, for example, Henriques *et al* 1976). However, there will be gradient terms in the free energy of the type

$$F_G = \frac{7\zeta(3)}{4\pi^2}(k_B T_c)^{-2}\frac{\hbar^2\rho}{8m^2}\partial_\mu d^*_{ia}\partial_\mu d_{ia}, \tag{18}$$

with i and a colour and spin indices, μ a space index, and ρ the total mass density. For densities close to the critical density for quark matter to be formed, and for systems of radius greater than about 10^{-13} cm, the contribution of the tensor potential to the free energy dominates that of the gradient terms. Thus spin-resonance effects may be observable in the quark matter produced by heavy-ion collisions. It might also be possible to produce a sample of quark matter with a large enough radius for spin-resonance effects, by imploding a tiny quantity of ordinary matter using laser beams, though this is likely to be a thought experiment.

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Note added in proof. Since completing this work, we have received a Caltech preprint by S Frautschi (CALT-68-701) which refers to as yet unpublished work by Barrois on a relativistic treatment of superconducting quark matter.

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